Some basic integrals

Don't forget C

$$\int (af(x) + bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^k\,dx = \frac{1}{k+1}x^{k+1} + C \quad \text{for } k \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \cos x\,dx = \sin x + C \qquad \int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C \qquad \int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A \sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$$

$$\sin A \cos A = \frac{1}{2}\sin 2A$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

t-substitution

Let $t = \tan\frac{x}{2}$. Then

$$\tan x = \frac{2t}{1-t^2} \qquad dx = \frac{2}{1+t^2}\,dt$$

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

Integration techniques

Substitution $\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$

(Repeated use: Integrate layer by layer)

Trig. Substitution

For functions with $\sqrt{\text{quadratic}}$

For $\sqrt{a^2-x^2}$ try $x = a\sin\theta$

For $\sqrt{x^2-a^2}$ try $x = a\sec\theta$

For $\sqrt{a^2+x^2}$ try $x = a\tan\theta$

Completing square first if x term in quadratic $\neq 0$

eg $\int \frac{dx}{\sqrt{x^2+2x+3}} = \int \frac{dx}{\sqrt{(x+1)^2+2}}$

Let $x+1 = \sqrt{2}\tan\theta$, then

$$\sqrt{(x+1)^2+2} = \sqrt{2\tan^2\theta+2} = \sqrt{2\sec^2\theta} = \sqrt{2}\sec\theta$$

Rationalization

For functions with $x^{\frac{1}{n}}$ (Let $u = x^{\frac{1}{n}}$)

Integration of Rational functions

See page ④

Integration by parts $\int u\,dv = uv - \int v\,du$

For products of x^m, e^{ax}, sin or cos, ln, arcsin or arccos

See page ④

Integration of trig. functions

Product of sine, cosine $\xrightarrow{\text{formula}}$ **Sum**

eg $\int \sin 3x \cos 7x\,dx$, $\int \cos^2 x\,dx$

Formulas

$$\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

$$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$

$$\sin A\sin B = -\frac{1}{2}[\cos(A+B)-\cos(A-B)]$$

$A=B \Rightarrow$

$$\sin A\cos A = \frac{1}{2}\sin 2A$$

$$\cos^2 A = \frac{1}{2}(1+\cos 2A)$$

$$\sin^2 A = \frac{1}{2}(1-\cos 2A)$$

$\int \sin^m x \cos^n x\,dx$

- m odd: $\sin x\,dx = -d\cos x$ or let $u=\cos x$
  n odd: $\cos x\,dx = d\sin x$ or let $u=\sin x$

eg $\int \sin^2 x \cos^3 x\,dx$ (odd) $= \int \sin^2 x \cos^2 x \cos x\,dx$

$$= \int \sin^2 x\,(1-\sin^2 x)\,d\sin x$$

- m, n both even: Reduce degree by formulas

t-substitution

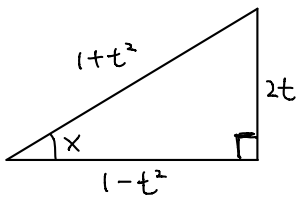
(Rational functions in $\sin x, \cos x$)

Let $t = \tan\frac{x}{2}$ Then $dx = \frac{2}{1+t^2}dt$

$$\tan x = \frac{2t}{1-t^2} \quad \sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2}$$

Rational functions in $\sin x, \cos x$ $\xrightarrow{t=\tan\frac{x}{2}}$ Rational function in t

(Then use technique for rational function)



(eg $\int \frac{dx}{4\sin x + 3\cos x + 3}$)

Integration of Rational Function $\frac{p(x)}{q(x)}$

- Decompose $\frac{p(x)}{q(x)}$ into partial fractions
- If $\deg p \geq \deg q$, do long division first

Terms appear in partial fractions:

- $\frac{A}{ax+b}$ or $\frac{A}{(ax+b)^k}$, $k>1$

  Easy to integrate: $dx = \frac{1}{a}d(ax+b)$

- $\frac{Ax+B}{ax^2+bx+c}$ or $\frac{Ax+B}{(ax^2+bx+c)^k}$,

  Trig. substitution, Completing square & Reduction formula

Formulas: $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\frac{x}{a} + C$

$\int \tan\theta\, d\theta = \ln|\sec\theta| + C$

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

Guideline for integration by parts on

$$\int x^n f(x)\,dx$$

- If $f(x) = \sin x$, $\cos x$, e^x then integrate $f(x)$, differentiate x^n
- If $f(x) = \arcsin x$, $\arccos x$, $\ln x$ then integrate x^n, differentiate $f(x)$

may need repeated integration by parts

eg $\int x^2\cos x\,dx = \int x^2\,d\sin x$

$\int x^2\ln x\,dx = \int \ln x\,\frac{1}{3}dx^3$

eg $\int e^x\cos 2x\,dx$

integrate e^x twice and rearrange terms